1. Details of Module and its structure

Module Detail	
Subject Name	Physics
Course Name	Physics 01 (Physics Part-1, Class XI)
Module Name/Title	Unit 5, Module 7, Angular Momentum Chapter 7, System of particles and Rotational motion
Module Id	Keph_10707_eContent
Pre-requisites	Kinematics, laws of motion, basic vector algebra
Objectives	After going through this module, the learners will be able to : • **Understand** angular momentum of a moving point object • **Recognize** angular momentum of a rotating rigid object • **Deduce c**onditions for conservation of angular momentum • **Apply** the law of conservation of angular momentum to understand daily life situations
Keywords	Angular momentum, conservation of angular momentum, Right hand palm rule, moment of momentum

2. Development Team

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1. UNIT SYLLABUS

Unit V: Motion of System of Particles and Rigid body

Chapter 7: System of particles and Rotational Motion

Centre of mass of a two-particle system; momentum conservation and centre of mass motion. Centre of mass of a rigid body; Centre of mass of a uniform rod.

Moment of a force; torque; angular momentum; law of conservation of angular momentum and its applications.

Equilibrium of rigid bodies; rigid body rotation and equations of rotational motion; comparison of linear and rotational motions.

Moment of inertia; radius of gyration; values of moments of inertia for simple geometrical objects (no derivation). Statement of parallel and perpendicular axes theorems and their applications.

2. MODULE-WISE DISTRIBUTION OF UNIT SYLLABUS 8 Modules

The above unit is divided into eight modules for better understanding.

Module 1	• Rigid body • Centre of mass • Distribution of mass • Types of motion: Translatory, circulatory and rotatory
Module 2	• Centre of mass • Application of centre of mass to describe motion • Motion of centre of mass
Module 3	• Analogy of circular motion of a point particle about a point and different points on a rigid body about an axis • Relation v = rω • Kinematics of rotational motion
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Module 5	• Torque • Types of torque • Dynamics of rotator motion • $m = I\alpha$
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Module 7

3. WORDS YOU MUST KNOW

Let us remember the words we have been using in our study of this physics course.

- Rigid body: An object for which individual particles continue to be at the same separation over a period of time.
- Point object: A point object is much smaller than the distance due to change in its position.
- Distance travelled: The distance an object has moved from its starting position. Its SI unit is m and it can be zero or positive.
- Displacement: The distance an object has moved from its starting position moves in a particular direction. Its SI unit is m and it can be zero, positive or negative.
- Speed: Rate of change of position and its unit m/s.
- Average speed: Average speed $=\frac{\text{Total path length}}{\text{Total time interval}}$ Its unit is m/s
- Velocity (v): Rate of change of position in a particular direction and its unit is m/s.
- Instantaneous velocity The velocity at an instant is defined as the limit of the average velocity as the time interval Δt becomes infinitesimally small velocity at any instant of time
- Uniform motion: object covers equal distance in equal interval of time.
- Non uniform motion: object covers unequal distance in equal interval of time.

- Acceleration (a): rate of change of velocity with time and its unit is m/s^2. Velocity may change due to change in its magnitude or change in its direction or change in both magnitude and direction.

- Constant acceleration: Acceleration which remains constant.
- Momentum (p): The impact capacity of a moving body is mv and its unit is kg m/s.
- Force (F): Something that changes the state of rest or uniform motion of a body. Unit of force is Newton. It is a vector, as it has magnitude which tells us the strength or magnitude of the force and the direction of force is very important.
- Constant force: A force for which both magnitude and direction remain the same with passage of time.

- Variable force: A force for which either magnitude or direction or both change with passage of time.

- External unbalanced force: A single force or a resultant of many forces that act externally on an object.

- Kinematics: Study of motion without involving the cause of motion.
- Dynamics: Study of motion along with the cause producing the motion.
- Vector: A physical quantity that has both magnitude and direction .displacement is a vector, force is a vector, acceleration is a vector etc.

- Vector algebra: Mathematical rules of adding, subtracting and multiplying vectors.

- Resolution of vectors: A vector can be resolved in two mutually perpendicular directions. We used this for vector addition and in our study of motion in 2 and 3 dimensions.

- Dot product: two vectors on multiplication yield a scalar quantity. Dot product of vector A and B: A.B= $|A||B|cos\theta$ where θ is the angle between the two vectors. Dot product is a scalar quantity and has no direction. It can also be taken as the product of magnitude of A and the component of B along A or product of B and component of A along B.

- Work: Work is said to be done by an external force acting on a body if it produces displacement W= F.S cosθ, where work is the dot product of vector F(force) and Vector S (displacement) and θ is the angle between them . Its unit is joule and dimensional formula is ML^2T^{-2}. It can also be stated as the product of component of the force in the direction of displacement and the magnitude of displacement. Work can be done by constant or variable force and work can be zero, positive or negative.

- Energy: The ability of a body to do work. Heat, light, chemical, nuclear, mechanical are different types of energy. Energy can never be created or destroyed it only changes from one form to the other.

- Kinetic Energy: The energy possessed by a body due to its motion = ½ mv^2, where 'm' is the mass of the body and 'v' is the velocity of the body at the instant its kinetic energy is being calculated.

- Conservative force: A force is said to be conservative if the work done by the force in displacing a body from one point to another is independent of the path followed by the particle and depends on the end points. Example: gravitational force.

- Non-conservative forces: **A** force is said to be non-conservative if: the work done by it on an object depends on the path and the work done by it through any round trip is not zero. Example: friction.

- Work Energy theorem: Relates work done on a body to the change in mechanical energy of a body i.e., $W = F.S = \frac{1}{2} mV_f^2 - \frac{1}{2} mV_i^2$

- Conservation of mechanical energy: Mechanical energy is conserved if work done is by conservative forces.

- Potential energy due to position: Work done in raising the object of mass m to a particular height (distance less than radius of the earth) = 'mgh'.

- Collision: Sudden interaction between two or more objects. We are only considering two body collisions.
- Collision in one dimension: Interacting bodies move along the same straight path before and after collision.

- Elastic collision: Collision in which both momentum and kinetic energy is conserved.
- Inelastic collision: Momentum is conserved but kinetic energy is not conserved.

- Coefficient of restitution: The ratio of relative velocity after the collision and relative velocity before collision. Its value ranges from 0-1.

- Torque: It is rotational analogous of Force and it has following characteristics:

  i. Torque is the turning effect of the force about the given axis of rotation.
  ii. The torque equals the moment of the force about the given axis/ point of rotation.

iii. Just as force equals the rate of change of linear momentum, torque equals the rate of change of angular momentum.

- Mechanical equilibrium: It implies either the object is at rest and stays at rest is said to be in static mechanical equilibrium or the center of mass, of the system, moves with a constant velocity is said to be in dynamic mechanical equilibrium.

- First condition for equilibrium: is that the center of mass of the body has zero acceleration; this happens when if the vector sum of all external forces acting on the body is zero. It is also called the condition for the translatory equilibrium. In vector and component forms, we can write:

$$\overrightarrow{F_{net}} = 0$$

- Second condition for an extended body to be in equilibrium is that the body must have no tendency to rotate. It is also called the condition for the rotational equilibrium. In vector and component forms, we can write:

$$\overrightarrow{\tau_{net}} = 0$$

- Center of gravity: It is that point in a given body, around which the resultant torque, due to the gravity forces, vanishes. The concept can be useful in designing structures, especially buildings, bridges etc., so that they remain stable under the influence of the forces acting on them.

4. INTRODUCTION

We had seen, in the unit of laws of motion, how the linear momentum, of a moving particle can be regarded as to measure of its linear motion; we had also seen how the rate of change of linear momentum is related to the force. Likewise, the concept of Angular momentum is used in rotational motion. All moving objects can have some sort of angular momentum but it is used most often to describe rotating objects. Forces in certain cases can change the angular momentum of an object. As in the case of linear momentum, the greater the mass of an object and the faster it is moving, the more difficult it is to slow it down or to stop its movement as well as rotation.

In the module on torque, we studied that a body when subjected to an external torque undergoes angular acceleration i.e. change in its angular velocity. Also, the change in angular velocity depends on the moment of inertia of the system if the torque acting on the system and its time duration, remain same. In such a case, higher the moments of inertia lower the change in angular velocity. Therefore, torque is intricately related with the moment of inertia and angular velocity of a given system. The quantity is directly related to the moment of inertia of a system, and its angular velocity, is its angular momentum.

How do we define angular momentum? Just as the moment of a force is the rotational analogue of force, the quantity, angular momentum is the rotational analogue of linear momentum. We shall first define angular momentum for the special case of a single particle. Like the moment of a force, angular momentum is also defined through a vector product. It is also called moment of (linear) momentum.

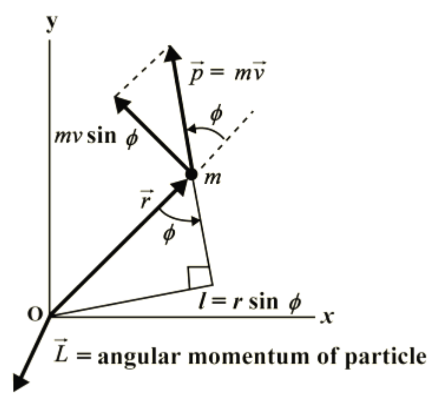
For a particle having a constant mass m, a velocity$\vec{v}$, a momentum $\vec{\mathbf{p}}$ and a position vector $\vec{\mathbf{r}}$, relative to the origin O, of an inertial frame, we define the angular momentum as:

$$\vec{\boldsymbol{\ell}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$$

The value of $\vec{\boldsymbol{\ell}}$ depends on the choice of origin O, since it involves the particle's position vector relative to O.

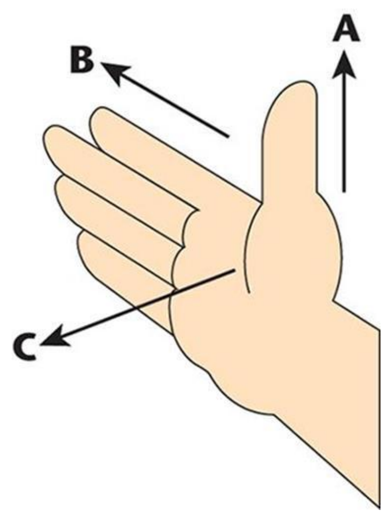
The SI units of angular momentum *are kg m*^{2}/s.

The direction of angular momentum vector is perpendicular to the plane defined by the of position vector ($\vec{\mathbf{r}}$) and the linear momentum $\vec{\mathbf{p}}$.



The Right-Hand Palm rule, for vector products, can be used for determining its direction.

According to it, if $\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$, the direction of $\vec{\mathbf{A}}$ is represented by the thumb of the stretched right palm and the fingers represent the direction of $\vec{\mathbf{B}}$, the direction of their cross product $\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is represented by the direction along which the palm faces, as shown below.



It means, from equation,

$$\vec{\boldsymbol{\ell}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$$

that if the direction of $\vec{\mathbf{r}}$, the position vector is represented by the thumb of the stretched right palm, and the fingers represent the direction of $(\overrightarrow{\mathbf{p}})$ the linear momentum, the direction of their cross product $(\overrightarrow{\boldsymbol{\ell}})$,the angular momentum is represented by the direction in which the palm faces.

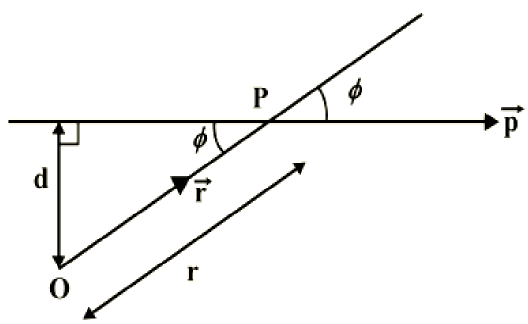
The magnitude of angular momentum can be written as

$$\boldsymbol{\ell} = \mathbf{r\ p\ sin\varnothing}$$

Where, Ø is the angle between $\vec{\mathbf{r}}$ **and** $\vec{\mathbf{p}}$.

This means that the magnitude of angular momentum can also be expressed as:

$$\boldsymbol{\ell} = \mathbf{p.d}$$

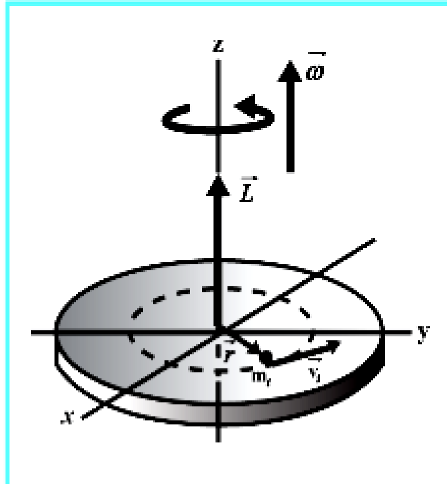


Where, $\mathbf{d = r\ sin\varnothing}$ is the length of the perpendicular, drawn from O, on the line drawn along the direction of $\overrightarrow{\mathbf{p}}$.

This distance plays the role of "lever arm" for the momentum vector.

5. ANGULAR MOMENTUM OF A RIGID BODY

Consider a rigid object rotating about a fixed axis, that coincides with the z axis of a coordinate system, (as shown in the Figure). Let's determine the angular momentum of this object. Each particle of the object rotates in the x-y plane about the z axis, with an angular speed ω. The magnitude of the angular momentum, of a particle of mass m_i about the z axis is



$$\ell_i = m_i v_i r_i.$$

As $v_i = r_i \omega$, we can express the magnitude of the angular momentum of this particle as

$$\ell_i = m_i r_i^2 \omega$$

The direction of angular momentum for this i^{th} particle is directed along the z axis. We can now find the angular momentum (which in this situation has only a z component) of the whole object by taking the sum of ℓ_i over all the particles:

$$\boldsymbol{\ell}_{\mathbf{net}} = \sum_{i=1}^{n} \mathbf{m_i r_i^2 \omega} = \boldsymbol{\omega} \sum_{i=1}^{n} \mathbf{m_i}\, r_i^2 = \mathbf{I}\, \boldsymbol{\omega}$$

Here, I= $\sum_{i=1}^{n} \mathbf{m_i r_i^2}$, is the moment of inertia of the object about the z axis.

6. RELATION BETWEEN ANGULAR MOMENTUM AND TORQUE

When a net force acts on a particle, its velocity and momentum change; so its angular momentum may also change. We can show that the rate of change of angular momentum is equal to the torque of the net force. We take the time derivative of Eq. (1), using the rule for the derivative of a product:

$$\frac{d\vec{\ell}}{dt} = \frac{d(\vec{r}\ \mathrm{X}\, \vec{p}\,)}{dt} = \frac{d\vec{r}}{dt}\ \mathbf{x}\, \vec{p} + \ \vec{r}\ \mathbf{x}\ \frac{d\vec{p}}{dt}$$

The first term is zero because it contains the vector product of two vectors that are really, the same vector. In the second term we can replace $\frac{d\vec{p}}{dt}$ with the net force. Hence,

$$\frac{d\vec{\ell}}{dt} = \vec{r} \times \vec{F} = \vec{\tau} \qquad \left(\text{as } \vec{F} = \frac{d\vec{p}}{dt}\right)$$

Hence, the rate of change of angular momentum of a particle equals the torque of the net force acting on it.

7. ANGULAR MOMENTUM AND TORQUE FOR A SYSTEM OF PARTICLES:

The total angular momentum of a system of particles about some point axis, is defined as the vector sum of the angular moment of the individual particles:

$$\overrightarrow{\ell_{net}} = \overrightarrow{\ell_1} + \overrightarrow{\ell_2} + \cdots \ldots + \overrightarrow{\ell_n}$$

Where, the vector sum is over all n particles in the system.

Differentiating the above equation with respect to time gives the net torque as:

$$\frac{d\overrightarrow{\ell_{net}}}{dt} = \frac{d(\overrightarrow{\ell_1} + \overrightarrow{\ell_2} + \cdots \ldots + \overrightarrow{\ell_n})}{dt} = \overrightarrow{\tau_1} + \overrightarrow{\tau_2} + \cdots \ldots + \overrightarrow{\tau_n} = \overrightarrow{\tau_{net}}$$

We conclude that the total angular momentum of a system can vary with time only if a net external torque is acting on the system. It applies even if the center of mass is accelerating, provided the torque and angular momentum are evaluated relative to an axis through the center of mass.

Let us do some problems to understand the use of these concepts in different situations**:**

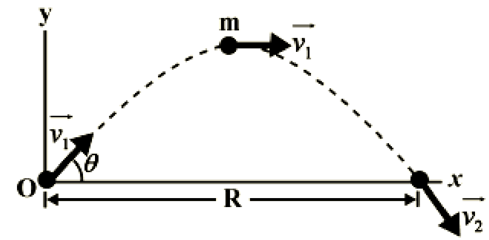
EXAMPLE:

A projectile of mass m is launched with an initial velocity v at an angle θ with the horizontal as shown in figure.

Find the angular momentum of the particle about the origin just before it strikes the ground.

SOLUTION:

As shown in the figure above a particle is projected with velocity v at an angle θ with the horizontal. It's speed we know will be same when it strikes the ground, also the angle with ground will be equal to that of angle of projection but the vertical component of velocity will get reversed in direction.



Therefore,

Initial velocity $\vec{v}_{\text{initial}} = \mathbf{v\,cos\theta\,\hat{i} + v\,sin\theta\,\hat{j}}$

Final velocity $\vec{v}_{\text{final}} = \mathbf{v\,cos\theta\,\hat{i} - v\,sin\theta\,\hat{j}}$

Angular momentum of the projectile, considering it to be point sized, about the origin will be given by:

$\vec{\ell} = \vec{r}\ \mathbf{X}\,\vec{p}$ …………………… (1)

here $\vec{r}$ **is the position vector of** the **point where** the **projectile hits the ground** and $\vec{p}$ is the linear momentum of the projectile when it hits the ground.

As the range of the particle can be written as $\mathbf{R} = \frac{\mathbf{v^2\,sin2\theta}}{\mathbf{g}}$

Hence

$\therefore \vec{r} = \mathbf{x\,\hat{i} + y\,\hat{j}} = \frac{\mathbf{v^2\,sin2\theta}}{\mathbf{g}}\,\mathbf{\hat{i} + 0\,\hat{j}}$ ………………… .(2)

$\vec{p} = \mathbf{m}\ \vec{v}_{\text{final}} = \mathbf{m\,(v\,cos\theta\,\hat{i} - v\,sin\theta\,\hat{j})}$ ………. (3)

Using equation (1), (2) and (3) one can write

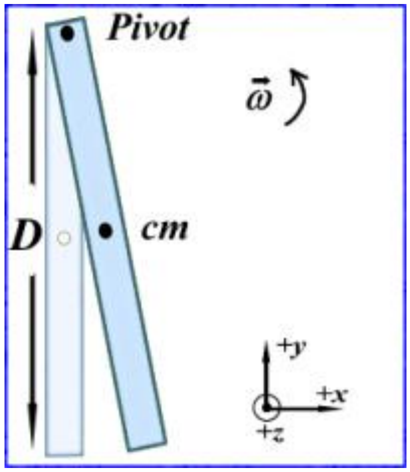
$$\vec{\ell} = \left(\frac{\mathbf{v^2\,sin2\theta}}{\mathbf{g}}\right)\mathbf{\hat{i}} \times [\mathbf{m\,(v\,cos\theta\,\hat{i} - v\,sin\theta\,\hat{j})}]$$

$$\therefore \vec{\ell} = \frac{-V^3 \sin 2\theta}{g} \sin\theta\, \hat{k}$$

EXAMPLE:

A rod of mass m and length D is pivoted about one end. As shown in the figure, the rod can rotate in the x-y plane, counterclockwise, as viewed from the +z- axis, with an angular speed ω.

Find the angular momentum of the rod about the 'pivot-point'.



SOLUTION:

Let I cm be the moment of inertia, of the rod, about an axis passing through the rod's center of mass and perpendicular to the x-y plane, Also let I pivot be the moment of inertia about an axis passing through the pivot point perpendicular to the x-y plane.

First method:

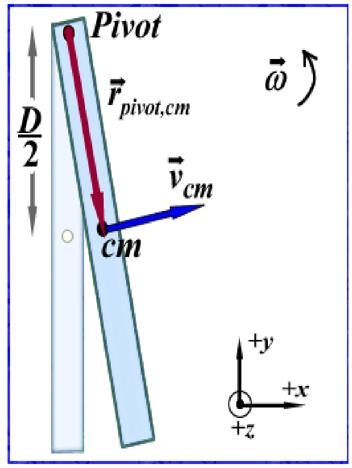
The rod rotates about a fixed axis passing through the pivot point. The motion of the rod is contained in the xy-plane, perpendicular to the axis of rotation. The angular momentum about the axis passing through the pivot is:

$\overrightarrow{L_{pivot}} = I_{pivot}\, \vec{\omega}$.......(1)

Where, I_{pivot} is the moment of inertia of the rod about the axis passing through the pivot point. It, being the moment of inertia of a rod, about its end point which can be given as

$\therefore \overrightarrow{L_{pivot}} = (\frac{mD^2}{3})\vec{\omega}$ -------------(1)

Another method:



We can also think that the rod is rotating and translating, we can then calculate the angular momentum of the rod about the pivot point as the sum of the angular momentum centre of mass of the rod about the pivot point and angular momentum of the rod about the centre of mass. Hence,

$$\overrightarrow{L_{pivot}} = \overrightarrow{r_{pivot,cm}} \times m\overrightarrow{v_{cm}} + I_{cm}\,\overrightarrow{\omega}.... \qquad (2)$$

Where $\vec{r}_{pivot,cm}$ is the position vector of the rod's center of mass with respect to the pivot point and $\vec{v}_{cm}$ is the velocity of the center of mass.

We can now show that the results, given by equation (1) and equation (2) are equivalent to each other.

1. The axis, passing through the pivot point and through the center of mass, are parallel axis and are separated a distance *D/2*. The parallel axis theorem implies that:

$$I_{pivot} = I_{cm} + m(D/2)^2 \quad \text{.................} \quad (3)$$

2. The center of mass is the equivalent point of the rigid object therefore, as for any other point of the rod, it rotates about the pivot with the angular speed ω. The linear speed of the center of mass can be expressed as:

$$v_{cm} = \omega D/2 \quad \text{....................} \quad (4)$$

3. The position and velocity vectors are perpendicular to each other:

$$\vec{r}_{pivot,cm} \perp \vec{v}_{cm}$$

As a result,

$$\vec{r}_{pivot,cm} \times \vec{v}_{cm} = |\vec{r}_{pivot,cm}||\vec{v}_{cm}| \sin 90^0 \hat{k} = (D/2)\vec{v}_{cm}\ \hat{k} \text{........} (5)$$

Starting with equation (2) and using equation (5) we obtain:

$$\overrightarrow{L_{pivot}} = \vec{r}_{pivot,cm} \times m\vec{v}_{cm} + I_{cm}\vec{\omega} = (D/2)mv_{cm}\,\hat{k} + I_{cm}\omega\,\hat{k}$$

Using (4) the above expression becomes:

$$\overrightarrow{L_{pivot}} = (m\left(\frac{D}{2}\right)^2 + I_{cm})\omega\,\hat{k} \ldots\ldots(6)$$

From equation (3) and (6), we can write

$$\overrightarrow{L_{pivot}} = I_{pivot}\,\vec{\omega}$$

This is the same as the result given by equation (1).

8. CONSERVATION OF ANGULAR MOMENTUM

We all know that we see the sun appear to 'go up' and 'come down' in the sky every day. The spinning of the earth, around its axis, helps us understand this beautifully. The mere thought, of the earth's stopping its rotation abruptly during the night, seems preposterous and unthinkable. But why can't it stop?

If there is no net external torque acting on a system, or if the system is isolated,

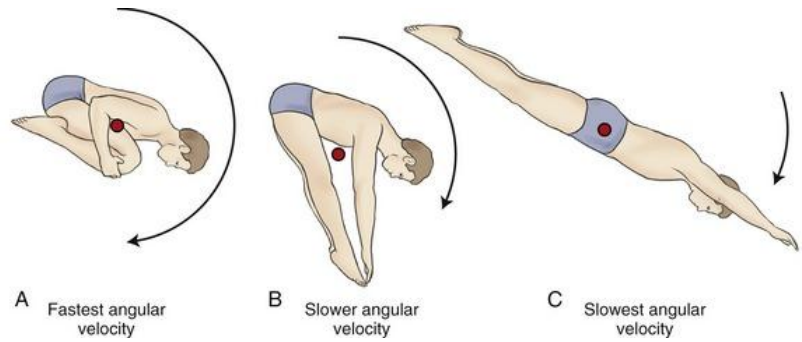
$$\overrightarrow{\boldsymbol{\tau}_{\mathbf{net}}} = \frac{\mathbf{d}\overrightarrow{\boldsymbol{\ell}_{\mathbf{net}}}}{\mathbf{dt}} = \mathbf{0}$$

i.e $\overrightarrow{\boldsymbol{\ell}_{\mathbf{net}}} = \mathbf{constant}.$

This means that there is no change in angular momentum, i.e. the angular momentum of the system remains conserved, when it is isolated, or when the net external torque, acting on it, is zero.

If the system is an object rotating about a fixed axis ***I*** is its moment of inertia about this axis. In this case, we can express the conservation of its angular momentum by saying that:

$$\vec{l} = \mathbf{I_1}\overrightarrow{\boldsymbol{\omega_1}} = \mathbf{I_2}\overrightarrow{\boldsymbol{\omega_2}}$$



Here, I_1 and I_2 are the moments of inertia of the (isolated) system, for two different arrangements of its 'mass- particles'. Its corresponding angular velocities, in the two cases, are ω_1 *and* ω_2.

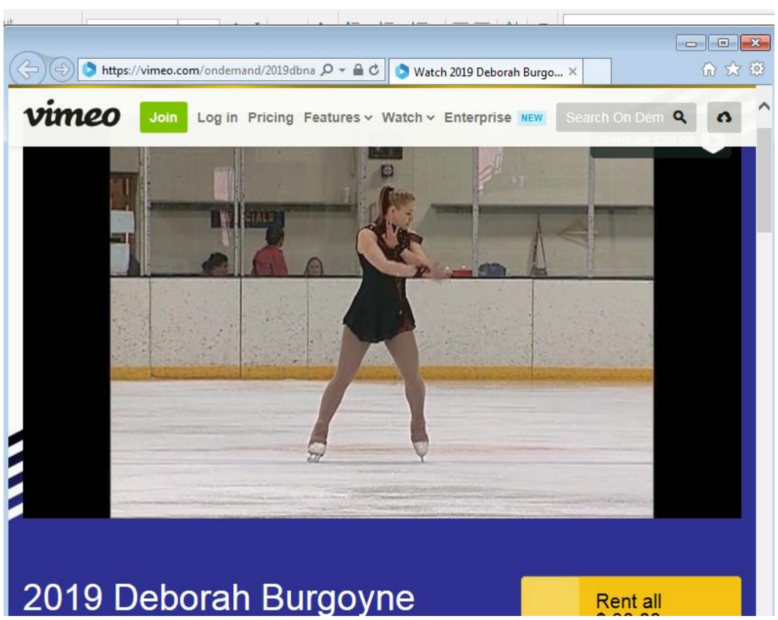
This result is valid both for rotation about a fixed axis and for rotation about an axis through the center of mass of a moving system as long as that axis remains parallel to itself.

We only require only that the net external torque acting on the system is zero.

Acrobats, skaters, divers and other sports persons make use of the principle of conservation of angular momentum to show off their feats. At the time of jumping, the diver gives herself/ himself a slight rotation, by which he/she acquires some angular momentum. When he/she is in air, there is no torque acting on him and therefore his angular momentum must be conserved. If he folds his body (fig A) to decrease his/her moment of inertia his rotation must become faster. If he unfolds his/her body before entering into pool of water, his/her moment of inertia increases and he/she must rotate slowly (fig C).

https://live.staticflickr.com/35/106394787_633237da39_b.jpg

Similarly the angular speed of the skater increases when the skater pulls his/her hands and feet close to his/her body, thereby decreasing moment of inertia. If one neglects friction between skates and ice, no external torques act on the skater. The change in angular speed is due to the fact that, because angular momentum is conserved, the product $I\,\omega$ remains constant, and a decrease in the moment of inertia of the skater causes an increase in the angular speed.



https://vimeo.com/ondemand/2019dbnainv

watch the video clip

The above idea is also used by **cats, dogs, horses and most animals** that wish to jump roll run with different speeds make use of the principle of conservation of angular momentum.

NOTE:

1. **The resultant torque acting on an object about an axis through the center of mass equals the time rate of change of angular momentum regardless of the motion of the center of mass. This theorem applies even if the center of mass is accelerating, provided the torque and angular momentum are both evaluated relative to the center of mass.**
2. **Energy, linear momentum, and angular momentum of an isolated system all remain constant or conserved.**
3. **When there is no net external torque acting on a system, then** $\overrightarrow{\tau_{net}} = \frac{d\overrightarrow{\ell_{net}}}{dt} = 0$

   i.e $\overrightarrow{\ell_{net}} = \text{constant}$. **This means there is no change in angular momentum, i.e. the angular momentum of the system remains conserved under this condition.**

EXAMPLE:

Suppose a girl sitting on a swivel chair (which can rotate), shown in Figure given below, is spinning at 0.800 rev/ s with her arms extended. She has a moment of inertia (girl and chair) of 2.34 kg · m^2 with her arms extended and of 0.363 kg · m^2 with her arms close to her body.

(a) What is her angular velocity in revolutions per second after she pulls in her arms?

(b) What is her rotational kinetic energy before and after she does this?

SOLUTION:

As there is no external torque on the girl, the movement of her arms results in a change in her angular velocity as per the law of conservation of angular momentum.

(a) Let I and I′ respectively are initial and final moment of inertia of the girl and chair taken together. While ω and ω′ be the initial and final angular velocity respectively of this system. $I\omega = I'\omega'$ …….(1)

We are given that: $I = 2.34\ \mathrm{kg \cdot m^2}$

$\omega = 0.800$ rev/ s

$I' = 0.363\ \mathrm{kg\ m^2}$

Substituting these values in equation (1) we get

$\omega' = I\,\omega / I'$

$$\omega' = \frac{2.34 \times 0.800}{0.363} = 5.16 \text{ rev/s}$$

(b) Rotational kinetic energy is given by $K.E._{rot} = \frac{I\omega^2}{2}$

The initial value is found by substituting known values into the equation, after converting the angular velocity to rad/s:

$\omega = 0.800 \times 2\pi$ rad/s

Therefore, initial K.E$_{.rot}$ = $\frac{\mathbf{I\omega^2}}{\mathbf{2}} = \frac{\mathbf{2.34 \times (0.800 \times 2\pi)^2}}{\mathbf{2}}$ = 29.6 j

The final rotational kinetic energy is $KE'_{rot} = \frac{\mathbf{I'\omega'^2}}{\mathbf{2}}$

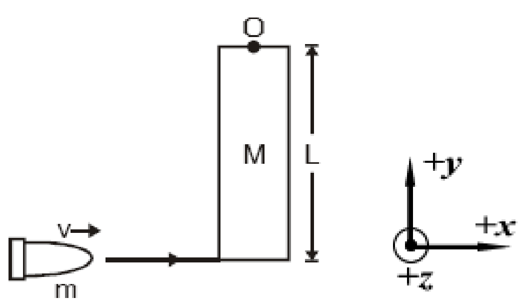
Substituting known values of I′ and ω′ = 5.16 x 2π rad/s into this equation..

$$KE'_{rot} = \frac{\mathbf{I'\omega'^2}}{\mathbf{2}} = \frac{\mathbf{0.363 \times (5.16 \times 2\pi)^2}}{\mathbf{2}} = 191 \text{ J}$$

We see that the final angular velocity as well as the final kinetic energy, are much greater than their initial values. Can one think of the reason behind the increase in kinetic energy of the system? This increase in rotational kinetic energy comes from the work done by the girl in pulling in her arms.

EXAMPLE:

A rod of length L and mass M is hinged at point O. A small bullet of mass m hits the rod, as shown in the figure. The bullet gets embedded in the rod. Find the angular velocity of the system just after the impact.



SOLUTION:

The net torque on the (rod-bullet) system, at the time of collision between the bullet and the rod is zero. Hence, the law of conservation of angular momentum can be applied here. As the system will rotate about point O we need to calculate angular momentum about O. We have,

$$\vec{\ell}_{\text{initial}} = \vec{\ell}_{\text{final}}$$

$$\vec{\ell}_{\text{initial}} = \vec{\ell}_{\text{rod}} + \vec{\ell}_{\text{bullet}}$$

As the rod is at rest initially its angular momentum is zero while for the bullet we will use

$$\vec{\ell} = \vec{r} \times \vec{p}$$

$$\vec{\ell}_{\text{initial}} = 0 + \mathrm{L\,m\,v}\sin 90^{0}\,\hat{k} = \mathrm{Lmv}\,\hat{k} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The final angular momentum of the system can be written as

$$\vec{\ell}_{\text{final}} = (\mathrm{I}_{\text{rod}} + \mathrm{I}_{\text{bullet}})\,\vec{\omega}$$

The moment of inertia of both the rod and the bullet are to be measured about point O; the two move together with a common angular velocity ω.

For the moment of inertia of rod, we will use the parallel axis theorem:

$$\mathrm{I}_{\text{rod}} = \mathrm{I}_{\text{CM}} + \mathrm{m\,r}^2$$

$$\mathrm{I}_{\text{rod}} = \frac{\mathrm{M\,L}^2}{12} + \mathrm{M}\left(\frac{\mathrm{L}}{2}\right)^2 = \frac{\mathrm{M\,L}^2}{3}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Also, $\mathrm{I}_{\text{bullet}} = \mathrm{m\,L}^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$

Using equation (1), (2) and (3), we get

$$\mathrm{Lmv}\,\hat{k} = \left(\frac{\mathrm{M\,L}^2}{3} + \mathrm{m\,L}^2\right)\vec{\omega}$$

Therefore $\vec{\omega} = \dfrac{\mathrm{mv}\,\hat{k}}{\left(\frac{\mathrm{M}}{3} + \mathrm{m}\right)\mathrm{L}}$

Try Yourself:

1. **If the polar caps of earth suddenly melt, will the day length of the day be affected?**

2. **A person sits near the edge of a circular platform revolving with a uniform angular speed. What will be the change in the motion of the platform, if the person starts moving from the edge towards the centre of the platform?**
3. **Comet move around the sun in highly elliptical orbits. As a result, the distance from the sun changes significantly while it moves. Does its angular speed get affected due to these changes in its distance from the sun?**

9. SUMMARY

1. The angular momentum, of a system of n particles about the origin, is given by:

$$\overrightarrow{\ell_{net}} = \sum_{i=1}^{n} \vec{r_i} \times \vec{p_i}$$

Here, $\vec{r_i}$ is the position vector of ith particle whose linear momentum is given by $\vec{p_i}$

For a rigid body the angular momentum, about an axis can be expressed as:

$$\overrightarrow{\ell_{net}} = I\vec{\omega}$$

2. For a rigid body rotating about a fixed axis (say, z-axis) of rotation, $\ell_z = I\omega$, where I is the moment of inertia about z-axis. In general, the angular momentum ℓ for such a body is not along the axis of rotation. Only when the body is symmetric about the axis of rotation, L is along the axis of rotation. In that case, $\ell = \ell_z = I\omega$.

3. The angular acceleration of a rigid body rotating about a fixed axis is given by $I\alpha = \tau$.

4. If the external torque τ acting on the body is zero, its angular momentum about the fixed axis (say, z-axis), ℓ_z $(=I\omega)$ remains constant.